Thus in Fig. 1, in which the system of axes $O$–$x$, $y$, $z$ is represented in orthographic projection,[1] is shown the point $P_0$, whose projection on the plane $xOy$ is $P_0'''$, the vertical $P_0L$, whose projection on the plane $xOy$ is $P_0'''L$, the latitude $\phi = \measuredangle P_0'''LP_0$, the meridian plane $LP_0P_0'''$ (also shown by its traces $m_2$, $m_3$), the longitude $\lambda = \measuredangle T_xMP_0'''$, the horizontal plane $NESW$ (also shown by its traces $s_1$, $s_2$, $s_3$), the north-and-south line $NP_0S$ (coincident with the line connecting the intersection of $s_2$ and $m_2$ with that of $s_3$ and $m_3$), the east-and-west line $EP_0W$ (coincident with the line connecting the intersections $n_1s_1$ and $n_2s_2$, where $n_1$ and $n_2$ are traces of the plane which passes through $P_0$ and is parallel to $xOy$).

*A level surface* is an equipotential surface of the field of force referred to above, *i. e.*, a surface, fixed with respect to the axes $O$–$x$, $y$, $z$, at each point of which the normal is the vertical. A *curve of constant astronomical latitude* is a locus of points, on a level surface, for which $\phi$ is constant. In particular, *the equator* is the locus for which $\phi = 0$. A *curve of constant astronomical longitude* is a locus of points, on a level surface, for which $\lambda$ is constant.[2] The east-and-west line at a general point $P_0$ is *not* tangent to the curve of constant latitude which passes through $P_0$. A curve at each point of which the tangent is the east-and-west line we shall call an *east-and-west curve*.[3] It is not difficult to see that the east-and-west curves lie on level surfaces and also in planes which are perpendicular to the earth's axis of rotation.[4] Hence the one-parameter family of east-and-west curves, which lie on a particular level surface, are cut from this level surface by the one-parameter family of planes which are perpendicular to the earth's axis of rotation.

§ 2. **Terms in Optics.** If rays of light meet the surface of a body, the surface is said to be illuminated in the region where the rays meet it. The appearance of brightness of an illuminated surface depends upon the quantity of light which this surface sends to the retina of the eye. The quantity of light which an element of surface sends to the eye depends on the *intensity of illumination* of this surface element and on the reflecting power, or *albedo*, of the surface. For parallel ray illumination (such as that due to the sun or other distant sources of light) the *intensity of illumination* of a plane surface element, whose normal makes an angle $\epsilon$ with the direction of the rays of light, is given by the formula

$$B = L' \cos \epsilon,$$

where $L'$ is the intensity of illumination of a plane surface which cuts the rays of light orthogonally.

Of the quantity of light which falls upon the surface of a body, a portion is

[1] This figure is correctly drawn according to the rules of axonometry. $T_xT_yT_z$ is the axonometric triangle, $m$ and $s$ are the axonometric traces of the planes $[m_2, m_3]$ and $[s_2, s_3]$ respectively. Therefore in the plane of the drawing the line $s$ is perpendicular to the line $P_0L$ and the line $m$ is perpendicular to the line $s_3$.

[2] The above definitions are practically the same as those given by Pezzetti, *Trattato di Geodesia Teoretica* (1905), § 5.

[3] Similarly, *north-and-south curves* might be defined. In general, such curves are different from the curves of constant longitude.

[4] This is a consequence of the fact, stated above, that the east-and-west line at $P_0$ is the intersection of the horizontal plane at $P_0$ by the plane which passes through $P_0$ and is perpendicular to the earth's axis. For, the horizontal plane at $P_0$ is the tangent plane at $P_0$ to the level surface which passes through $P_0$, and the plane through $P_0$ and perpendicular to the axis, is its own tangent plane, and in general the tangent to the curve of intersection of two surfaces is the intersection of the tangent planes to the two surfaces.

absorbed, another portion is reflected and still another portion may be transmitted through the body. Only that portion which is reflected contributes to the brightness. The manner of the reflection depends upon the nature of the surface. A reflecting surface is said to be *perfectly smooth* when corresponding to each incident ray there is but a single reflected ray, the reflection occurring according to the law that the angles of incidence and reflection are equal and lie in the same plane; that is, *the normal to the surface at the point of reflection bisects the angle formed by the incident and reflected rays.* A reflecting surface is said to be *perfectly rough or dull* when it scatters the incident light which illuminates it in all directions and appears equally bright from all directions of view. The *brightness* of a surface element $f$ is defined as the quantity of light which this surface element sends to the retina of the eye, divided by the area of the retinal image of $f$. For parallel ray illumination the *brightness* of a surface element is given by the formula

$$H = AL' \cos \epsilon,$$

where $A$ is the albedo of the surface and $L'$ and $\epsilon$ are quantities already defined. The brightness is independent of the direction from which the surface is viewed, provided the surface is perfectly dull.

For a surface which is not plane, the quantity $H$ varies from point to point, but has constant values along certain curves of the surface. These curves, provided $L'$ and $A$ are constants, are the loci of points of the surface for which the angle $\epsilon$, between the surface normal and the direction of the rays of light, has constant values. They are called *lines of constant intensity of illumination or curves of constant brightness.* The particular curve of this kind for which $\epsilon = 90°$, $\cos \epsilon = 0$, is called *the line of shade.*

Now let us consider a perfectly smooth reflecting surface as defined above. Such a surface has no brightness at any of its points, except at those isolated points from which emanate those unique rays of reflection which pass to the observer's eye. These points are very brilliant and hence are called *brilliant points.*[1]

*Loci of Brilliant Points.* The two ideal types of reflecting surface which are defined above may be regarded as extremes between which there lies another type which is worthy of consideration. For the perfectly smooth reflecting surface there corresponds to each incident ray but a *single reflected ray,* which emanates from the point where the incident ray pierces the surface, while for the perfectly dull reflecting surface there corresponds to each incident ray the *two-parameter family of reflected rays* which emanate from the point where the incident ray pierces the surface. For a type of reflecting surface which has already been considered by the author,[2] there corresponds to each incident ray the *one-parameter family of reflected rays* which emanate from the point $P$ where the

[1] The above definitions and formulas are those given by Weiner, *Lehrbuch der Darstellenden Geometrie,* in the chapters on Beleuchtungslehre, Vol. I, p. 390, Vol. II, p. 200.

[2] See the introduction to the author's paper, "Brilliant Points of Curves and Surfaces," *Trans. of the American Math. Society,* Vol. 9, No. 2, pp. 245–279.

incident ray pierces the surface and form a cone of revolution whose vertex is $P$ and whose axis is a tangent line to the surface at the point $P$. Such a surface may be realized[1] by scratching a perfectly smooth reflecting surface along the members of a one-parameter family of curves of the surface. The surface of a piece of metal can be thus scratched by holding a piece of emery cloth against

FIG. 2.

it while it rotates in a lathe. Whereas perfectly smooth reflecting surfaces, which are illuminated, exhibit isolated *bright points* (defined above as brilliant points), surfaces of this type exhibit *bright curves*. See the photograph, Fig. 2. These curves are the loci of points which the author has defined as *brilliant points of curves*.[2]

According to the law of reflection stated on page 72 and to the suggestion made in the last footnote, we have the following

*Definitions.* A point $P$ is said to be a brilliant point of the surface $t$ (or of the curve $c$, plane or twisted) with respect to the source (of light) $P_1$ and the recipient (an observer's eye) $P_2$ when the internal bisector of the angle $P_1PP_2$ is the normal to the surface $t$ (or a normal to the curve $c$) at the point $P$. In

[1] Of course, only approximately, as are also the other two types.

[2] In order to get the concept of "brilliant point of a curve" and to be able to see that there corresponds to each incident ray a cone of reflected rays, let us think of the scratches as forming ridges and furrows which are perfectly smooth, and furthermore let us think of each ridge or furrow as belonging to a tube-surface. As the cross-section of a tube-surface approaches zero, a brilliant point of the surface approaches a point of the curve which is approached by the vanishing tube-surface. This consideration suggests the definition of brilliant point of a curve which is here given. Those rays, reflected from a perfectly smooth tube-surface, which correspond to incident rays which lie in a plane, form a ruled surface. As the cross-section of the tube-surface approaches zero, the narrow band of incident rays approaches a single ray, and the ruled surface of reflected rays approaches one nappe of a cone of revolution, of which the vertex is the point $Q$ where the incident ray meets the curve approached by the vanishing tube-surface, the axis is the tangent at $Q$ to this limit curve and the angle is the same as that which the incident ray makes with the axis of this cone extended in the direction opposite to that on which this nappe lies.

particular $P_1$ and $P_2$ may be infinitely distant. As a consequence of these definitions the following theorem is immediately evident.

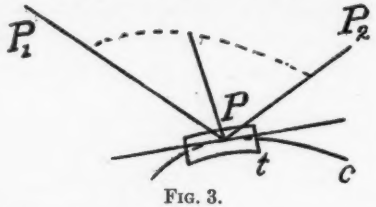


FIG. 3.

THEOREM. *The brilliant points of a surface (or a curve) are those of its points at which it is tangent to ellipsoids of revolution of which $P_1$ and $P_2$ are the foci. If, in particular, $P_1$ and $P_2$ are infinitely distant the confocal ellipsoids are replaced by parallel planes.*

If we denote by $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$, the direction cosines of the directed lines $PP_1$ and $PP_2$ respectively, and by $l$, $m$, $n$ numbers proportional to the direction cosines of the normal to the surface $t$ (or the tangent line to the curve $c$) at the point $P$, then the condition that $P$ should be a brilliant point, with respect to $P_1$ and $P_2$, of the surface is

$$(1) \qquad (l_1 - l_2)l + (m_1 - m_2)m + (n_1 - n_2)n = 0, \qquad \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0;$$

and of the curve $c$ is[1]

$$(2) \qquad (l_1 + l_2)l + (m_1 + m_2)m + (n_1 + n_2)n = 0.$$

§ 3. **The Analogies.** Now that we have defined all of the necessary terms, we are able to state what the interpretations really are to which reference was made above. In order to do this let us think of a level surface of the earth's weight-field of force as being (1) a perfectly dull reflecting surface, (2) a reflecting surface of the type last defined in § 2 and scratched along the members of its one-parameter family of east-and-west curves.

We then have the following theorems:

THEOREM I. *The curves of constant astronomical latitude on a level surface are also the curves of constant brightness of this surface, regarded as a perfectly dull reflecting surface, when the source of illumination is an infinitely distant light in the direction of the earth's axis of rotation. In particular, the equator is the line of shade.*

THEOREM II. *The curves of constant astronomical longitude on a level surface are also the loci of the brilliant points of the one-parameter family of east-and-west curves of this surface, with respect to infinitely distant sources and recipients.*[2]

[1] For more details see the author's paper already referred to.

[2] That is, if a level surface of the earth, supposed metallic, were scratched along its east-and-west curves, an observer on the moon, or any other distant body in a general direction, would see as a curve of light some curve of constant astronomical longitude, the curve being the assemblage of the images of the sun, which may be in any general direction, in the reflecting scratches.

The truth of theorem I follows immediately from the definitions of "brightness" and "astronomical latitude." The first of these quantities is proportional to cos $\epsilon$ and the second, $\phi$, is equal to $90° - \epsilon$, since the rays of illumination are parallel to the earth's axis. Hence the locus of points on a level surface for which $\epsilon$ is constant, is in one case a curve of constant brightness and in the other, a curve of constant astronomical latitude.

Theorem II may be proved geometrically by means of the special case of the theorem stated in § 2. According to this special case, a point $P$ of a curve $c$ is a brilliant point of $c$ with respect to the infinitely distant points $P_1$ and $P_2$, if $c$ is tangent at $P$ to one of the single infinity of planes which are perpendicular to the line $b$ which bisects internally the angle $P_1PP_2$. If, in particular, the curve $c$ is an east-and-west curve of a level surface, it lies in a plane perpendicular to the earth's axis of rotation $a$. Therefore, in order to find a brilliant point of an east-and-west curve $c$, it is only necessary to determine a point $P$ of $c$ at which the tangent is parallel to the line $d$ in which the plane of $c$ is cut by a plane perpendicular to the line $b$. The plane $\mu$ which is normal to an east-and-west curve $c$ at a point $P$ of $c$, and therefore also parallel to the earth's axis $a$, is the astronomical meridian plane of $P$. In the plane of each of the single infinity of east-and-west curves of a level surface there is a line $d$, a point $P$ at which the tangent to the east-and-west curve $c$ which lies in this plane is parallel to $d$, and a corresponding plane $\mu$. All of the lines $d$ are parallel (being perpendicular to both $a$ and $b$) and hence all of the planes $\mu$ are parallel and consequently all of the points $P$ have the same astronomical longitude. Therefore the locus of the brilliant points $P$ is a curve of constant astronomical longitude.

In order to get an analytic proof of Theorem II, let us represent by $\omega$ the angular velocity of the earth's rotation, and by $U$ the function $f(x, y, z)$ which is defined by the integral

$$U = \int_V \frac{dm}{\rho}, \qquad \rho = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2},$$

where $dm$ represents that element of mass of the earth which is situated at the point whose coördinates, with respect to the axes $O$–$x$, $y$, $z$, are $x_i$, $y_i$, $z_i$, the integration being extended throughout the whole volume of the earth. Then the potential function of the earth's weight-field of force is[1]

$$W = U + \frac{\omega^2}{2}(x^2 + y^2).$$

The equations of the level surfaces are of the form

$$W(x, y, z) = C, \tag{1}$$

where $C$ has constant values. For the particular level surface which passes

[1] See Pezzetti, *loc. cit.*, § 2.

through the point $P_0$, $(x_0, y_0, z_0)$, $C = W(x_0, y_0, z_0)$. The direction cosines of the vertical at $P_0$ are proportional to[1]

$$W_x^\circ, \quad W_y^\circ, \quad W_z^\circ.$$

Since the direction cosines of the earth's axis (here coincident with $Oz$) are

$$0, \quad 0, \quad \pm 1,$$

the equation of the astronomical meridian plane at $P_0$ is

$$W_y^\circ(x - x_0) - W_x^\circ(y - y_0) = 0, \tag{2}$$

and hence the astronomical longitude $\lambda$ at $P_0$ is given by the relation

$$\tan \lambda = \frac{W_y^\circ}{W_x^\circ}, \tag{3}$$

and the differential equations of the two-parameter family of east-and-west curves are[2]

$$\frac{dx}{W_y} = \frac{dy}{-W_x} = \frac{dz}{0}. \tag{4}$$

Since in our particular problem the source $P_1$ and the recipient $P_2$ are infinitely distant, the quantities $l_1, m_1, n_1, l_2, m_2, n_2$ are constants, as are also the quantities

$$\alpha = l_1 + l_2, \quad \beta = m_1 + m_2, \quad \gamma = n_1 + n_2,$$

which are proportional to the directional cosines of the bisector $b$ defined in the geometric proof of theorem II. If now we replace $l, m, n$ by $W_y, -W_x, 0$ respectively, condition (2), § 2, takes the form

$$\alpha \cdot W_y - \beta \cdot W_x = 0 \quad \text{or} \quad \frac{W_y}{W_x} = \frac{\beta}{\alpha}. \tag{5}$$

This equation is the locus of the brilliant points, with respect to the infinitely distant points $P_1$ and $P_2$, of the two-parameter family of east-and-west curves (4). In its second form we recognize it also (by virtue of relation (3)) as the locus of points which have the same constant astronomical longitude $\lambda = \tan^{-1}(\beta/\alpha)$. Therefore the curve of intersection of this locus (5) with a level surface is both *a curve of constant astronomical longitude* on this level surface and *a locus of brilliant points*, with respect to the infinitely distant points $P_1$ and $P_2$, *of the one-parameter family of east-and-west curves* on this level surface. It is evident

[1] The symbols $W_x$, $W_y$, $W_z$ are used for the partial derivatives $\partial W/\partial x$, $\partial W/\partial y$, $\partial W/\partial z$ respectively, and the symbols $W_x^\circ$, $W_y^\circ$, $W_z^\circ$ stand for the values which these derivatives have at the point $x = x_0$, $y = y_0$, $z = z_0$.

[2] These differential equations may be written in the form: $W_x\,dx + W_y\,dy + W_z\,dz = 0$, $dz = 0$, from which it is evident that the solution is $W = C$, $z = k$, where $C$ and $k$ are constants. From this we see (what has already been stated in § 1) that the east-and-west curves are the intersections of the level surfaces with the planes perpendicular to the earth's axis.

that we can choose the directions of the infinitely distant points $P_1$ and $P_2$ (in an infinite variety of ways) so as to make the expression $\lambda = \tan^{-1}(\beta/\alpha)$ for the astronomical longitude have any value we wish.[1]

From the form of the function $W$, it is evident that the level surfaces (1) are, in general, *not* surfaces of revolution. If, in particular, the distribution of the earth's matter were such that the function $U$ assumed the form $f(v, z)$, where $v = \sqrt{x^2 + y^2}$, then the level surfaces would be surfaces of revolution, the east-and-west curves would be identical with the curves of constant astronomical latitude, and the curves of constant astronomical longitude would be the meridian curves[2] of these surfaces of revolution.

Hence Theorem II yields the following corollary.

COROLLARY. *The locus of the brilliant points, with respect to an infinitely distant source $P_1$ and an infinitely distant recipient $P_2$, of the one-parameter family of curves which are cut from a surface of revolution by planes perpendicular to the axis, is a meridian curve of this surface of revolution.*[3]

The photograph on page 73 shows two such curves (due to two sources of light) on a brass sphere which has been properly scratched with emery cloth while rotating in a lathe. In reality, the sources and the recipient (observer's eye) are not infinitely distant, but they are practically so.

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## A THEOREM IN THE MODERN PLANE GEOMETRY OF THE ABRIDGED NOTATION.

BY ROBERT E. BRUCE, Boston University.

NOTE BY THE EDITORS. The following paper is another contribution fulfilling the spirit of the editorial in the January issue. While the subject matter is strictly elementary, the methods are elegant and forceful and should furnish an inspirational study for those interested in modern geometry. Those who may wish to read an introduction to the abridged notation will find chapter IV of Salmon's *Conic Sections* helpful.

**Introduction.** In common with polar reciprocation, projection, and certain other methods of modern geometry, the abridged notation may be used either for the proof of a theorem presupposed to be true or for the discovery of new theorems the exact form of which is unsuspected until the proof is complete. For either purpose an identity in terms of the abridged notation is selected and any number of algebraic operations performed upon it leading to a final identity. The first identity corresponds to the hypothesis, the last to the conclusion, and the intervening transformations to the proof. If the form of the theorem is

[1] It is also evident that the locus of the billiant points, with respect to a source $P_1$ and a recipient $P_2$, of any one-parameter family of curves on a surface $\sigma$, passes through the brilliant points, with respect to $P_1$ and $P_2$, of $\sigma$.

[2] A meridian curve of a surface of revolution is a curve which is cut from this surface by a plane which passes through the axis of rotation.

[3] If a surface of revolution were scratched along its meridian curves, the locus of the brilliant points would be a more complex curve. Such a locus has been considered by the author for the case of a sphere, in Vol XX, No. 10, p. 299 of this MONTHLY.

already suspected the identity of the hypothesis and the transformations are, as a matter of course, so selected as to bring the desired result. If, on the other hand, one desires to discover some theorem for the satisfaction of the discovery rather than for the theorem itself, the identity of hypothesis and the transformations may be selected almost at random; the work when done being interpreted in whatever geometry one chooses. In the following, the main theorem illustrates this latter use of the notation while the lemma proved in connection with corollary 8 illustrates the former.

**Proof of Main Theorem.** Let the $k$'s be constants other than zero and let the $S$'s be expressions of the same degree in the variables.

If

$$k_1S_1 + k_2S_2 \equiv k_3S_3 + k_4S_4$$

then

$$k_1S_1 - k_3S_3 \equiv k_4S_4 - k_2S_2$$

and

$$k_1S_1 - k_4S_4 \equiv k_3S_3 - k_2S_2.$$

Limiting the interpretation for simplicity to the case in point geometry where there are but two variables and where the $S$'s are of the second degree we have the following:

If a conic through the intersections of conics $S_1$ and $S_2$ passes through the intersections of conics $S_3$ and $S_4$, then there is a conic through the intersections of $S_1$ and $S_3$ that passes through the intersections of $S_4$ and $S_2$. Similarly for $S_1$, $S_4$ and $S_3$, $S_2$. Hence the following:

**Theorem.** *If it is possible to divide four given conics into two groups of two each in such a way that the eight points of intersection within the groups are on a conic, then there will be a conic through the eight points of intersection in each of the other arrangements of the conics into two groups of two each.*

It is obvious that various other theorems may be derived from the proof by changing the degree of $S$, the number of variables, or the particular geometry used.

The following are some of the corollaries of the theorem as stated:

**Corollary 1.** Consider three intersecting conics with their nine pairs of opposite chords of intersection. Denote the conics by $S_1$, $S_2$, and $S_3$ respectively. Denote the pairs of opposite chords of intersection as follows: One pair for $S_1$ and $S_2$ by $C_{12}'$, a second pair by $C_{12}''$, and the third by $C_{12}'''$. So for $S_1$, $S_2$ and $S_2$, $S_3$. Any one of these pairs of chords is a degenerate conic. Consider the four conics $S_1$, $C_{13}'$, $S_2$, and $C_{23}'$ grouping them as follows: $(S_1, C_{13}')$; $(S_2, C_{23}')$. The eight intersections within the groups are all on the conic $S_3$. Hence there exist two eight-point[1] conics,—one through the intersections of $S_1$ with $S_2$ and of $C_{13}'$ with $C_{23}'$, and the other through the intersections of $S_1$ with $C_{23}'$, and of $S_2$ with $C_{13}'$. But the group[2] $(S_1, C_{13}')$ may, aside from the above, be associated severally

[1] For sake of brevity the phrase "eight-point conics" will be used to denote the conics involved in the conclusion of the main theorem.

[2] This word is of course used in a non-technical sense.

with the groups $(S_2, C_{23}'')$ and $(S_2, C_{23}''')$, the hypothesis of the theorem being fulfilled in each case. Hence for the group $(S_1, C_{13}')$ there exist six eight-point conics derived through its association with $S_2$ and the various pairs of opposite chords of intersection of $S_2$ and $S_3$. But the group $(S_1, C_{13}')$ may also be associated with the group $(C_{23}', C_{23}'')$, since these two degenerate conics intersect on $S_3$, thus giving two more eight-point conics. Similarly for the groups $(C_{23}', C_{23}''')$ and $(C_{23}'', C_{23}''')$. Hence there exist 12 eight-point conics for the various groupings above, using $(S_1, C_{13}')$ as one pair of conics. But in place of $(S_1, C_{13}')$ there may be substituted any one of the following pairs of conics: $(S_1, C_{13}'')$, $(S_1, C_{13}''')$, $(C_{13}', C_{13}'')$, $(C_{13}', C_{13}''')$, $(C_{13}'', C_{13}''')$. Each substitution gives, of course, 12 new eight-point conics by taking severally the various groups we have used above with $(S_1, C_{13}')$. Hence there exist 72 eight-point conics for the cases considered. But just as $S_1$ and $S_2$ with the chords used intersect upon $S_3$, so also do $S_1$ and $S_3$ with their chords intersect upon $S_2$, and $S_2$ and $S_3$ upon $S_1$. Hence there exist 216 eight-point conics through the various intersections. The above may be summarized as follows:

*Given any three intersecting conics with their chords of intersection; there exist* 216 *different conics each containing eight of the points of intersection of the conics and chords with one another.*

This corollary has been worked out in detail for the sake of clearness. A moment's consideration, however, makes it plain that the result may be found at once from the following product: $2 \times {}_4C_2 \times {}_4C_2 \times 3 = 216$.[1] It is also evident that the same method may be applied to the cases where the hypothesis gives four or more intersecting conics with their chords of intersection.

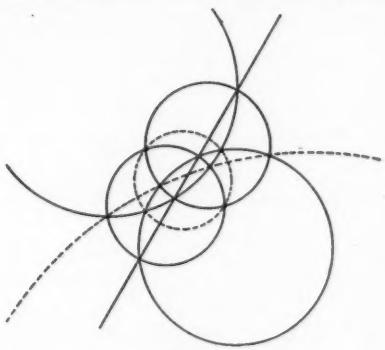

FIG. 1.

**Corollary 2.** If two pairs of conics have segments of the same two straight lines as opposite chords of intersection, the sixteen other points of intersection of the conics with each other are in two groups of eight each on two additional conics.

The two chords form a degenerate conic. And by hypothesis the four conics

[1] Where ${}_nC_r$ is the number of combinations of $n$ things taken $r$ at a time.

intersect upon it in two groups of two each. The conclusion, then, follows at once from the main theorem. The next two corollaries are special cases of the above.

**Corollary 3.** If two pairs of circles have different segments of the same line as their common chords, then the eight other points of intersection of the circles with each other are in groups of four each on two other circles. (See Fig. 1.)

The line at infinity is the second chord satisfying the hypothesis of corollary 2. The missing intersections are the circular points at infinity. Hence the conics of the conclusion are circles.

**Corollary 4.** Consider two quadrilaterals with different segments of the same two lines as diagonals. (See Fig. 2.) The pairs of opposite sides of the quadrilaterals are degenerate conics. Thus the hypothesis of corollary 2 is fulfilled. Hence there are two eight-point conics through the sixteen points of intersection of the sides of one quadrilateral with the sides of the other. This corollary is the basis of the construction of the following corollary.

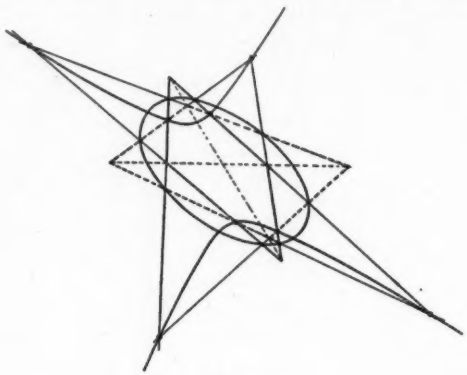

FIG. 2.

**Corollary 5.** *Construction of the conic through five given points.*[1]

Number the points 1, 2, 3, 4, 5. Let line 24 be $\alpha'$, 35 be $\beta'$, 25 be $b'$, and 34 be $a'$. Draw any two convenient lines, $a$ and $\beta$, through 1.

Let $Y$ be the intersection of $a$ and $a'$, $W$ of $a$ and $b'$, $I$ of $\beta$ and $\alpha'$, and $III$ of $\beta$ and $\beta'$. Let the line $WI$ be $d$, and $Y\,III$ be $d'$.

Let $X$ be the intersection of $d$ and $a'$, $II$ of $d$ and $\beta'$, $IV$ of $d'$ and $\alpha'$, and $Z$ of $d'$ and $b'$.

Let line $XZ$ be $b$, and $II\,IV$ be $\alpha$.

Let 6 be the intersection of $b$ and $\beta$, 7 of $a$ and $\alpha$, 8 of $b$ and $\alpha$.

The points 6, 7, 8 are on the conic through 1, 2, 3, 4, 5. Other sets of points may be found by varying the arbitrary lines $a$ and $\beta$.

[1] No diagram is drawn because the construction will be better appreciated by the reader if he draws the figure step by step. A figure is given in connection with a briefer statement of the construction added below.

*Proof of the construction.* The conic $\alpha\beta$ intersects the conic $\alpha'\beta'$ on the conic $dd'$. The conic $ab$ intersects the conic $a'b'$ on the conic $dd'$. Hence the intersections of $ab$ with $\alpha\beta$ and of $a'b'$ with $\alpha'\beta'$ are on a conic. But these points of intersection are the points 1, 2, 3, 4, 5, 6, 7, 8. Hence 6, 7, and 8 are on the conic through 1, 2, 3, 4, 5.

The following briefer statement of the method of construction has been found to satisfy the proof in all cases tested. (See Fig. 3.)

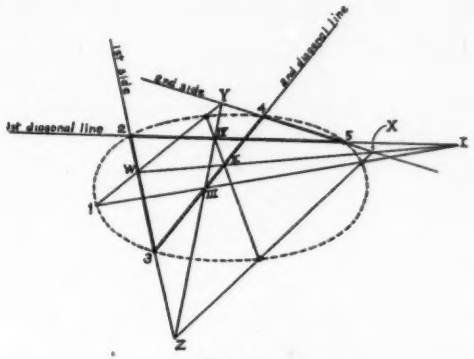


FIG. 3.

Selecting four of the given points, draw either pair of opposite sides and the two diagonals of the quadrilateral[1] of which these points are the vertices. Draw any two convenient lines through the fifth point.[2] Let the first of these lines intersect the first[3] and second diagonals in points $I$ and $III$ respectively. And let the second line intersect the first and second sides in points $W$ and $Y$ respectively. Call the points where the line $Y\,III$ meets the first side and the first diagonal $Z$ and $IV$ respectively; and the points where the line $W\,I$ meets the second side and the second diagonal $X$ and $II$ respectively. The following intersections are on the conic: $I\,III$ with $X\,Z$, $X\,Z$ with $II\,IV$, $II\,IV$ with $Y\,W$. By varying the lines through the fifth point as many points as desired may be found.

**Corollary 6.** *Pascal's Theorem:* "The intersections of the opposite sides of a hexagon inscribed in a conic are collinear." (See Figure 4.)

Let $ABCDEF$ be the given hexagon. Draw $C'F'$ any line that meets the conic, and form the two inscribed quadrilaterals $ABCF$ and $C'DEF'$. The degenerate conic formed by the lines $BC$ and $AF$ intersects the conic of the lines $AB$ and $CF$ on the given conic. Similarly for conics $C'D$, $EF'$ and $C'F'$, $ED$. Hence these four conics obey the hypothesis of the main theorem and by pairing

[1] That is, the ordinary convex quadrilateral of the elementary euclidean geometry,—not the complete quadrilateral. If no three of the points are in the same straight line a selection of this character is always possible.

[2] In the figure this point is 1.

[3] The numbering throughout is arbitrary.

the conics as follows: $BC \cdot AF$ with $EF' \cdot C'D$ and $AB \cdot CF$ with $C'F' \cdot ED$ the points 1, 2, 3, 4, 5, 6, 7, 8 are seen to be on a conic. But if the line $C'F'$ coincides with the line $CF$, the points 6, 7, 1, 2, 5 are collinear on the line $CF$. Hence the

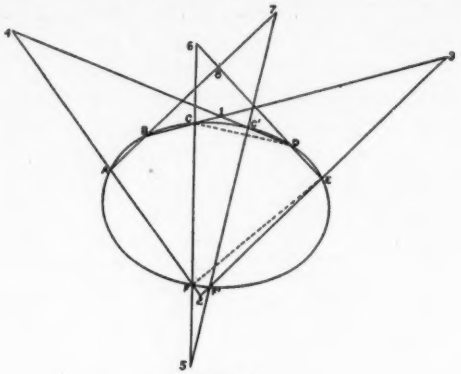

FIG. 4.

eight-point conic is degenerate and the points 4, 8, 3 are collinear on a line other than $CF$. But these points are the intersections of the opposite sides of the hexagon.

**Corollary 7.** "If three conics have a common chord the opposite common chords are concurrent." (*Salmon's Conic Sections*, Art. 266.) See Figure 5.

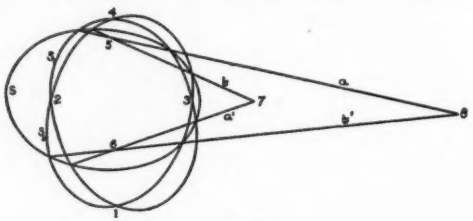

FIG. 5.

If the chords $a$ and $a'$ are associated with $S_1$, and $b$ and $b'$ with $S_2$, the two pairs of conics thus obtained obey the hypothesis of the main theorem, since the intersections within the groups are on $S$. Hence by associating chords $a \cdot a'$ with chords $b \cdot b'$ and conic $S_1$ with conic $S_2$ the points 1, 2, 3, 4, 5, 6, 7, 8 are seen to be on a conic. If $b$ coincides with $a$ the three conics have a common chord and the points 2, 3, 5, 7, 8 are collinear upon it. Hence the eight-point conic is degenerate and the points 4, 6, 1 are collinear. 6 is thus the point of concurrency of the three opposite chords of intersection. It is obvious that the proof may be applied to any three conics having a common chord.

**Corollary 8.** "If two conics have double contact with a third, their chords of contact with the third and a pair of their chords of intersection with each other are concurrent." (*Salmon's Conic Sections*, Art. 263.)

For the proof of this and other corollaries the following lemma is used in addition to the main theorem.

*Lemma.* If a conic intersects itself or is tangent to itself, it is degenerate and consists of two straight lines through the points of intersection or tangency.

If $S = 0$ is the equation of a conic, and $\alpha = 0$ and $\beta = 0$ the equations of straight lines, and $k$ a constant other than zero; then $S + k\alpha\beta = 0$ may be used to represent any conic intersecting $S$ or tangent to $S$, the particular relationship depending on the relationship of $\alpha$ and $\beta$ to $S$. (*Salmon's Conic Sections*, Arts. 249, 251.)

Since by hypothesis $S$ is to intersect itself or to be tangent to itself, $S \equiv m(S + k\alpha\beta)$, where $m$ is a constant other than zero.

Hence

$$S \equiv \frac{mk}{1 - m}\alpha\beta.$$

Now $m \neq 1$ since otherwise, from the first identity, the product $\alpha\beta$ would be identically zero. Hence the locus $S$ is the same as the straight lines $\alpha$ and $\beta$ combined.

*Proof of corollary 8.* Let $S_1$ have double contact with $S$ and let $S_2$ be tangent to $S$ at $G$ and intersect it at $H$ and $K$. The following groups of conics fulfill the hypothesis of the main theorem: $(S_1, EF \cdot EF)$ and $(S_2, GH \cdot GK)$.

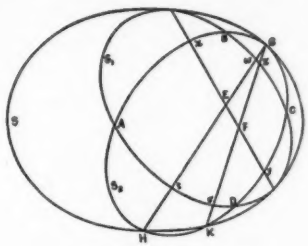


FIG. 6.

Hence there is a conic through the points $ABCD$ which is tangent to the line $HG$ at $E$ and to the line $KG$ at $F$. But if $F$ and $E$ coincide, this conic is tangent to itself. Hence it is degenerate consisting of the two lines $AC$ and $BD$ passing through the point $EF$. Moreover, under these conditions the conic $S_2$ has double contact with $S$. Apparently any conic having double contact with $S$ may be derived in this way and hence the corollary is proved.

**Corollary 9.** The second eight-point conic in figure 6 passes through the points $t$, $v$, $w$, $z$ and has double contact with $S_2$ at $x$ and $y$. Moreover, if $S_2$ has double contact with $S$, $w$ coincides with $z$ and $t$ coincides with $v$, and this second eight-point conic has double contact with $S_1$ also. Hence the following: If two conics have double contact with a third conic they also have double contact with a fourth conic, the chords of contact being different segments of the same two lines.

**Corollary 10.** If a pair of opposite chords of intersection of each of two conics with a third meet at the same point, that is, if the four lines are concurrent, a pair of opposite chords of intersection of the two conics with each other pass through the point.

Grouping each of the two conics with its given chords of intersection with the third conic, we have four conics obeying the hypothesis of the main theorem. Hence there is an eight-point conic through the intersections of the two conics with each other and the intersections of the chords with each other. But since the four points of intersection of the chords with each other are coincident the eight-point conic either intersects itself or is tangent to itself. Hence it is degenerate and the corollary is established.

It is apparent that corollary 8 is a special case of the above.

**Corollary 11.** If in corollary 2 the two straight lines are coincident we have the following: If different segments of the same straight line are chords of contact of two pairs of conics having double contact, then the sixteen points of intersection of the two pairs of conics with each other are in two groups of eight each on two additional conics.

Several special cases of this corollary follow.

**Corollary 12.** The four points of intersections of any two hyperbolas with each other are on a conic through the four points of intersection of their asymptotes with each other.

This follows directly from corollary 11 and from the fact that any hyperbola and its asymptotes have double contact, the chord of contact being the line at infinity. The other eight-point conic in this case gives the following:

**Corollary 13.** The four points of intersection of one hyperbola with the asymptotes of a second are on a conic through the four points of intersection of the second hyperbola with the asymptotes of the first.

**Corollary 14.** Given two pairs of concentric circles; the eight points of intersection of the circles with each other are in two groups of four each on two additional circles.

This follows from the fact that concentric circles have double contact on the line at infinity. Moreover the eight missing points of intersection are the circular points at infinity and thus the conics of the conclusion are circles.

**Corollary 15.** It is obvious that a hyperbola with its asymptotes and two concentric circles may also be used to discover two eight-point conics as in corollaries 12 to 14.

**Corollary 16.** Given a tangent to a parabola and two conics having double contact, the chord of contact being a segment of that line parallel to the axis of the parabola which passes through the point of contact of the tangent and parabola; the intersections of the parabola with either conic and of the tangent with the other are on a conic. The species of this conic is the same as the species of the given conic which meets it on the tangent. If this conic is a circle so is the resulting conic.

This corollary is seen to be true from the following considerations: The para-

bola has double contact with the tangent and the line at infinity, the parallel to the axis being the chord of contact. Hence corollary 11 may be applied to this case. The nature of the points of intersection of the given conics with the line at infinity fixes the species of the resulting conics since these points are on the resulting conics.

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## BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

*The Teaching of Arithmetic.* By DAVID EUGENE SMITH. Ginn and Company, Boston, 1913. v+196 pages.

*A Textbook on the Teaching of Arithmetic.* By ALVA WALKER STAMPER. American Book Company, New York, 1913. 284 pages.

Considerable attention should be paid to the teaching of arithmetic by all of those who have in charge the training of teachers and administrative officers of the public school system. Hence the universities as well as the normal and training schools should provide instruction along these lines. Both of these texts under discussion are well adapted for this purpose and it is to be hoped that they will enjoy wide use.

Eleven years ago Professor Smith and Professor McMurry of Teachers College prepared for the Teachers College Record an able article on the teaching of arithmetic. This was well adapted for instruction purposes and was so used until the edition was exhausted. In 1909 Professor Smith prepared for the same journal a new article with the same title and used some of the same material. This was also published in book form by Teachers College. The latter has been somewhat revised and expanded for this work from the press of Ginn and Co. The discussion of number games, briefly treated before, has been elaborated into a useful chapter. The chapter entitled "Subjects for Experiment" deserves the close attention of educators. Similar suggestions have been made in the MONTHLY for such experiments in connection with the high school and college mathematics. In this treatment some reference should certainly have been made to the tests for measurement of the efficiency of instruction in arithmetic, notably those suggested by S. A. Courtis. As a part of the recent examination of the New York City Schools such tests were used and in schools all over the country similar tests are being made. A discussion by Professor Smith of the value of these tests would have been a valuable and timely addition to the work.

The second of these works, by A. W. Stamper, takes up the topics much more in detail and does not treat the larger and more general phases of the subject. For this reason this book may be found more useful by normal schools and training classes. Some of the historical matter in this text is misleading. Thus it is quite incorrect to say that "the science of algebra originated in the fourth century B. C. in the Greek city of Alexandria in Egypt." Sciences do not originate in this definite and precise way and Alexandria is not connected with its origin

as intimately as Egypt and possibly India. Similarly trigonometry did not have its initial development in that city. A serious fault in this work seems to the reviewer to be the neglect to emphasize the decimal character of our number system and the import of this fact for the instruction in arithmetic. Constant emphasis on the decimal character of our number system, and use of it, greatly simplifies the work of computation and leads to a better understanding of the processes of reckoning. For instance, on page 50 it is suggested that in mentally adding, 85, 25, and 40, we should first add 15 to 85 making 100, then add the remaining 10 of 25, and finally add 40. This, it seems, is contrary to all well known methods of rapid mental computation which universally proceed from left to right. In this problem the child should be taught to first add 20 to 80, then add 40, and finally add the two remaining fives. These suggestions may seem to be trifles but it is precisely in these trifles that pupils obtain or lose comprehension of the processes of arithmetic.

Both authors recommend the so-called Austrian, or additive, method of subtraction but neither one presents the application of this system to problems in long division in which, instead of writing down the partial products, one writes only the successive *remainders*. Thus to divide 13650 by 243 the plan is as follows:

$$243)13650(56$$
$$150$$
$$42$$

$5 \times 3 = 15$; $5 + 0 = 5$. $5 \times 4 = 20$; $20 + 1$ (carried) $= 21$; $1 + 5 = 6$. $5 \times 2 = 10$; $10 + 2$ (carried) $= 12$; $12 + 1 = 13$. The 0, 5, 1 are set down in turn and form the remainder when $5 \times 243$ is subtracted from 1365. Proceeding in the same way the next product $6 \times 243$ is not set down but simply the remainder. In most European schools where the additive method is taught this form of long division is suggested. It is easily mastered and saves much of the *longness* of long division.

Both of these works are notable and desirable additions to the literature on instruction in arithmetic. Mathematicians have somewhat neglected this field and, partly in consequence, the erroneous notion has become rather widespread that the less an individual knows about mathematics the better qualified he is to write on arithmetic and instruction in arithmetic. These works present the material from the standpoint of the trained mathematician who is at the same time familiar with the elementary schools. Their influence will be felt in elementary instruction.

L. C. KARPINSKI.

*Analytic Geometry and Principles of Algebra.* By ALEXANDER ZIWET, Professor of Mathematics at the University of Michigan, and LOUIS ALLEN HOPKINS, Instructor in Mathematics at the University of Michigan. New York, The Macmillan Company, 1913. In the series edited by E. R. Hedrick. viii+369 pages and 150 figures.

As far back as the oldest inhabitant can remember it has been considered the proper thing to write text-books, and publishers are usually eager to assist the

unselfish missionary in presenting his views as to the proper manner of educating the masses. All too frequently the prospective author has responded to this high call before being kissed by the muse, and has been obliged to profit by the efforts of his predecessors to such an extent that each successive contribution secures an ever increasing degree of uniformity.

According to the historical researches of Heine, the massive wall was built about the city of Göttingen to prevent some enterprising docent from bringing new ideas into the university. To a large extent, so far as text-book writing is concerned, a very efficient wall of conservatism had been erected.

And then came a reaction; side by side with this conservative element a new faction was developed which represented the other extreme. We have algebras for boys, algebras for girls; geometry for engineers, geometry for medical students, and so on; probably we may soon expect trigonometry for red-headed students and even further differentiation.

Finally, once in a while we see the announcement of a text-book written by really chosen authors—persons of broad scholarship and ripe experience who undertake the task with a clearly defined purpose, having no axe to grind and fearless of charges of being too old or too new.

Since the book under review belongs distinctly to the third category, it demands more than a passing notice. The mechanical make-up of the book merits favorable comment; the large type, spaced lines, consistent use of italics, bold-face etc., the well executed figures, the full index, and the complete answers not only make for easy reading but put a big premium on neatness and orderliness on the part of the reader.

As the title indicates, the volume is concerned not only with analytic geometry, but also with the principles of algebra. Elementary algebra, plane and solid geometry and some knowledge of trigonometry are presupposed, but nothing of graphical processes nor of the subjects of advanced algebra.

The text begins much as the ordinary text on analytics, but the persistent use of $\Delta x = x_2 - x_1$ and the early introduction of determinants are innovations. Projection, polar coördinates, and vectors are all discussed before the equation of the straight line is mentioned. The projective forms of the equation are treated in great detail, that of one line being followed by that of a pencil, including the point and angle of intersection of two given lines. Here follows a second, more extensive development of determinants, including co-factors, minors, and elimination. Finally the normal form of the equation of a line is derived from the polar equation, and the usual applications made. Then follows a whole chapter on permutations and combinations, mathematical induction, and a systematic treatment of determinants, including multiplication.

In the chapter on the circle the ordinary treatment is enlivened by the use of determinants, the discussion of quadratic equations, inversion, pole and polar, pencils of circles and parametric representation. Since the quadratic equation may involve imaginaries, a full discussion of numbers, rational, irrational, fractional, complex, together with a geometric interpretation of each case is

given. This chapter closes with a proof of DeMoivre's theorem. A well written chapter on the properties and graphical representation of higher polynomials includes the concepts of the first and second derivatives, maxima and minima, points of inflexion, roots of numerical equations, both rational and irrational, Horner's method, and the expansion of a polynomial by Taylor's series. It comprises everything usually provided in the theory of equations, yet comes in so naturally that it seems a necessary part of analytic geometry.

The chapter on the parabola adds to the topics of the traditional course the parametric representation, area of a segment, and a full derivation of Simpson's rule for the approximate area under any curve, also an application to shearing force and bending moment.

The two chapters on central conics are more conservative; they are first discussed from the definition of the locus of a point the sum (or difference) of whose distances from two fixed points is a constant, and later by means of the eccentricity, in which it is shown that all conics can be cut from a right circular cone.

Then follows a chapter on algebraic curves, including the determination of the number of constants, the equations of tangents and normals, double points at the origin, and a detailed study of a number of particular algebraic and transcendental curves, incidentally developing the theory of logarithms. The principles of this chapter are applied to the study of empirical equations and a number of practical suggestions are added regarding cross-section paper, logarithmic paper, etc.

The part on solid geometry is very brief. The positive directions of the coördinate axes are so chosen that the $xy$-plane as viewed from the positive $z$-axis is the same as the $xy$-plane as previously studied. Linear systems of planes are treated fully, including the derivation of the volume of the tetrahedron determined by four non-coplanar points. In the discussion of the sphere we find inversion, tangent cone, polar theory and linear systems touched upon; the other quadric surfaces are but briefly treated, yet ruled surfaces are considered, and contour lines mentioned. Rotation of axes is all put in small type; it includes the derivation of Euler's formulas. A short appendix illustrates abridged multiplication and division.

Within the compass of this volume are found most of the subjects understood under the heading of a first course in analytic geometry, an almost complete treatment of advanced algebra, and a wealth of other material not found in either. And the book nowhere seems to be overloaded, but every page is lively and attractive.

The changes from one subject to another are not digressions, but are made consistently with the sound pedagogic principle that each idea is developed as needed, whether the idea comes from the box labeled algebra, or calculus, or what not.

A reviewer is always supposed to find some fault, just to show how much wiser he is than the authors of the book under review, but in this case the reviewer feels a certain embarrassment, and even detects symptoms of a wish to have

been the author himself. However, a few points may be mentioned. In discussing the bisectors of the angles between lines or between planes, the only method derived is valueless if one of the boundaries of the angle passes through the origin. Incidentally, does any book treat this subject properly? The whole discussion of positive and negative regions, so well done for curves, is hardly convincing when applied to lines and to planes. A considerable number of the later proofs are not complete, and in some cases rather too brief to be of great service, yet with proper skill on the part of the instructor may have influence in the right direction.

In those institutions in which advanced algebra is required for entrance, the book would either have to be abbreviated, or a number of topics repeated. Perhaps it is an argument to allow more alternatives for entrance, and make frank provision for teaching algebra to all in college.

VIRGIL SNYDER.

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## PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

### PROBLEMS FOR SOLUTION.

*Special Notice.*—Please reread the requests as to form of solutions on pp. 258–259 of the October 1913 issue. Unless these directions are observed by contributors, solutions must either be entirely rewritten by the committee or else rejected. Put all drawings on separate sheets.

MANAGING EDITOR.

#### ALGEBRA.

When this issue was made up no solutions had been received for numbers 401, 402, 404, and 406. Please give attention to these.

**405. Proposed by E. J. MOULTON, Northwestern University.**
Given the alternating series

$$s = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots.$$

(*a*) Let $s_n$ be the sum of the first $n$ terms of the series. Show that in order to make the difference $s - s_n$ numerically less than $1/2k$ ($k$ a positive integer) it is necessary and sufficient to take $n = k$; hence $s_{500}$ differs from $s$ by less than .001. (*b*) Let $s_n'$ be the sum of $s_n$ and $1/2$ the $(n+1)$th term of the series. Show that the difference $s - s_n'$ is numerically less than $1/2n(n+1)$; hence $s_{32}'$ differs from $s$ by less than .001.

**406. Proposed by S. A. COREY, Hiteman, Iowa.**
Solve the system of equations:

$$\begin{aligned}(1-x)(a_1 + a_2y + a_3z) &= d,\\ (1-y)(b_1 + b_2x + b_3z) &= g,\\ (1-z)(c_1 + c_2x + c_3y) &= h.\end{aligned}$$

**407. Proposed by E. B. ESCOTT, University of Michigan.**
In computing the values of the natural logarithms of 2, 3, and 5 by the following formulas:

$$\begin{aligned}\log 2 &= 2(7P + 5Q + 3R),\\ \log 3 &= 2(11P + 8Q + 5R),\\ \log 5 &= 2(16P + 12Q + 7R),\end{aligned}$$

where $P$, $Q$, and $R$ are numbers which were computed by infinite series (G. Chrystal, *Algebra*, Part II, chapt. 28), it is found, on comparing the results with the known values of these logarithms to 15 decimals, that there are the following errors: $-2{,}533$, $-4{,}052$, and $6{,}080$, respectively. Find the errors in $P$, $Q$ and $R$.

## GEOMETRY.

When this issue was made up no solutions had been received for numbers 410, 417, 421, and 427. Please give attention to these.

**434. Proposed by CLIFFORD N. MILLS, Bloomington, Ind.**

$ABC$ is any triangle with sides $a$, $b$, $c$. Prove by purely geometrical methods that the area $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = 1/2(a+b+c)$. (From Olney's *Geometry*, page 255, with slightly simplified notation.)

**435. Proposed by R. M. MATHEWS, Riverside, California.**

From a fixed point $A$ perpendiculars are dropped to the tangents drawn to a circle whose center is $O$. Prove that the locus of the feet of the perpendiculars is a limaçon.

**436. Proposed by A. J. KEMPNER, University of Illinois.**

Given in a plane two similar curves arbitrarily situated, except that they shall possess the same sense of direction (which, of course, does not mean that they are similarly located). Let corresponding points on both curves be joined by straight lines, and let all of these straight lines be divided in the same ratio $\lambda : 1$, $\lambda$ being any real number. Prove that the points of division all lie on a curve similar to the two given curves except when they all happen to fall together.

## CALCULUS.

When this issue was made up no solutions had been received for numbers 335, 337 to 340, 342, 348, and 350. Please give attention to these.

**356. Proposed by B. F. FINKEL, Drury College.**

A steel girder $l$ feet long and $w$ feet wide is moved on rollers along a passageway $a$ feet wide and into a corridor at right angles to the passageway. How wide must the corridor be to just admit the girder?

**357. Proposed by W. D. CAIRNS, Oberlin College.**

A continuous variable represented by a point on a vertical line changes according to such a law that it is reduced to $1/m$ of its value on being moved $a$ units upward, irrespective of the special position from which it is moved. Find the law of change, that is, the relation between the variable $y$ and the height $h$ of the variable point above a fixed point of the vertical line.

**358. Proposed by C. N. SCHMALL, New York City.**

About a given circle circumscribe the smallest parabola.

## MECHANICS.

When this issue was made up no solutions had been received for numbers 271 to 275 inclusive, and 277 to 279 inclusive. Also 268 and 269, proposed in 1912, have not been solved. Please give attention to these.

**288. Proposed by C. E. HORNE, Westminster College, Colorado.**

Show that the tangential velocity of a projectile at any point of its path is equal to the velocity it would have acquired in falling, under the influence of gravitation alone, from the directrix to the point in question (Hulburt's *Calculus*, p. 220, ex. 10).

**289. Proposed by C. N. SCHMALL, New York City.**

A particle of elasticity $e$ is projected with a velocity $v$ at an angle of elevation $\phi$ from a point on a smooth horizontal plane. Show that after $2v \sin \phi / g(1-e)$ seconds, it will cease to rebound and will move along the plane with a uniform velocity $v \cos \phi$.

### NUMBER THEORY.

When this issue was made up no solutions had been received for numbers 189, 191, 192, 196, 201, and 202. Please give attention to these.

**208. Proposed by E. T. BELL, Seattle, Washington.**
If an odd number is perfect, it cannot be the sum of two squares.

**209. Proposed by R. D. CARMICHAEL, Indiana University.**
Prove that the difference of the sixth powers of two integers cannot be the square of an integer

**210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.**
If $a$ and $b$ are relatively prime and $(a + b)$ is even, then $(a - b)ab(a + b) \equiv 0 \pmod{24}$.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

**394. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.**
Solve the equation $\sin^2 x \sin^2 2x = 5/16$.

SOLUTION BY R. M. MATHEWS, Riverside, California.

Taking the square root of each side of the given equation we obtain

$$(1)\ \sin x \sin 2x = \frac{\sqrt{5}}{4} \quad \text{and} \quad (2)\ \sin x \sin 2x = -\frac{\sqrt{5}}{4}.$$

From (1),

$$2 \sin^2 x \cos x = \frac{\sqrt{5}}{4}, \quad \text{or} \quad (3)\ \cos^3 x - \cos x + \frac{\sqrt{5}}{8} = 0.$$

Inspection shows that $-\frac{\sqrt{5}}{2}$ is a value of $\cos x$ in (3). Eliminating this impossible solution, we have the quadratic

$$(4) \qquad \cos^2 x - \frac{\sqrt{5}}{2} \cos x + \frac{1}{4} = 0,$$

the roots of which are $\cos x = \dfrac{\sqrt{5}+1}{4}$ and $\cos x = \dfrac{\sqrt{5}-1}{4}$.

The solutions of (2) are the negatives of those of (1) just found.

The side of a regular inscribed decagon is $(\sqrt{5} - 1)/2$, whence $(\sqrt{5} - 1)/4 = \cos 72°$. From this, it is easily shown that $(\sqrt{5} + 1)/4 = \cos 36°$.

Since in the original equations the functions are all squared, the signs which they may have in particular quadrants are immaterial. Hence

$$x = n\pi \pm \tfrac{1}{5}\pi, \quad x = n\pi \pm \tfrac{2}{5}\pi,$$

where $n$ is any positive or negative integer, are the real values satisfying the equation.

Also solved by C. E. GITHENS, HORACE OLSON, ALBERT R. NAUER, RICHARD MORRIS, A. M. HARDING, ELMER SCHUYLER, S. W. REAVES, W. C. EELLS, LEROY M. COFFIN, A. H. HOLMES, F. M. MORGAN, H. C. FEEMSTER, J. L. RILEY, H. E. TREFETHEN, and the PROPOSER.

**395. Proposed by V. M. SPUNAR, Chicago, Illinois.**
Solve the system of equations

$$x_1^2 x_2 = a_1, \quad x_2^2 x_3 = a_2, \quad x_3^2 x_4 = a_3, \quad \cdots, \quad x_n^2 x_1 = a_n.$$

I. Solution by Thomas E. Mason, Indiana University.

Assuming each $a$ different from zero and expressing each $x$ in terms of $x_1$ from the equations in order, we have

$$x_2 = \frac{a_1}{x_1^2},\ x_3 = \frac{a_2 x_1^4}{a_1^2},\ x_4 = \frac{a_3 a_1^4}{a_2^2 x_1^8},\ \cdots,\ x_i = \frac{a_{i-1}a_{i-3}^4 a_{i-5}^{16}\cdots}{a_{i-2}^2 a_{i-4}^8 a_{i-6}^{32}\cdots} x_1^{(-2)^i-1}.$$

Continuing this substitution we would obtain

$$x_{n+1} = \frac{a_n a_{n-2}^4 a_{n-4}^{16}\cdots}{a_{n-1}^2 a_{n-3}^8 a_{n-5}^{32}\cdots} x_1^{(-2)^n}.$$

But from the last of the given equations, we see that $x_{n+1}$ is defined to be $x_1$. Hence we have

$$x_1 = \frac{a_n a_{n-2}^4 a_{n-4}^{16}\cdots}{a_{n-1}^2 a_{n-3}^8 a_{n-5}^{32}\cdots} x_1^{(-2)^n}, \quad \text{or} \quad x_1^{2^n+(-1)^{n-1}} = \left(\frac{a_n a_{n-2}^4 a_{n-4}^{16}\cdots}{a_{n-1}^2 a_{n-3}^8 a_{n-4}^{32}\cdots}\right)^{(-1)^{n-1}}.$$

This can be solved by the ordinary trigonometric method and the $2^n + (-1)^{n-1}$ roots found. For each value of $x_1$ there will be corresponding values of $x_i$ found by substituting the value of $x_1$ in the formula above for $x_i$.

Solved in like manner by H. C. Feemster, A. H. Holmes, and S. W. Reaves.

II. Solution by F. C. Reisler and I. A. Barnett, University of Chicago.

Given the system of equations:

$$x_1^2 x_2 = a_1,\ x_2^2 x_3 = a^2,\ \cdots,\ x_n^2 x_1 = a_n.$$

If no $a = 0$, the following law is found to be true for all cases through $n = 4$.

$$x_k^{2^n-(-1)^n} = \prod_{l=0}^{n-1} a_{k+l}^{2^{n-1-l}(-1)^l}, \tag{1}$$

where $k = 1 \cdots n$, and $a_{n+q}$ is defined as $a_q$.

In order to show that (1) holds true for all values of $n$, we assume that it holds for $n = m$, that is,

$$x_k^{2^m-(-1)^m} = \prod_{l=0}^{m-1} a_{k+l}^{2^{m-1-l}(-1)^l}, \tag{2}$$

and prove that it holds for $n = m + 1$.

To do this we take a new set of $m + 1$ equations

$$y_1^2 y_2 = a_1,\ y_2^2 y_3 = a_2,\ \cdots,\ y_m^2 y_{m+1} = a_m,\ y_{m+1}^2 y_1 = a_{m+1}.$$

From these and the given equations we obtain the new system,

$$x_1^2 x_2 = y_1^2 y_2,\ x_2^2 x_3 = y_2^2 y_3,\ \cdots,\ x_m^2 x_1 = y_m^2 y_{m+1},\ a_{m+1} = y_{m+1}^2 y_1, \tag{3}$$

where $y_{m+1+p}$ is defined as $y_p$.

The solution of this system of equations is given by (2) as follows:

$$x_k^{2^m-(-1)^m} = \prod_{l=0}^{m-1} [y_{k+l}^2 y_{k+l+1}]^{2^{m-l-1}(-1)^l}.$$

In this product all of the factors except the first in the first couplet and the last in the last couplet become unity when combined in pairs, as can be seen by examining any such pair, for example,

$$(y_{k+l+1})^{2^{m-l-1}(-1)^l}(y_{k+l+1}^{2})^{2^{m-l-2}(-1)^{l+1}},$$

in which the second factor may be written $(y_{k+l+1})^{2^{m-l-1}(-1)^{l+1}}$, and hence the sum of the exponents is zero no matter whether $m$ is odd or even.

Therefore

$$x_k{}^{2^m(-1)^m} = y_k{}^{2^m} y_{k+m}{}^{(-1)^{m-1}} = \prod_{l=0}^{m-1} a_{k+l}{}^{(-1)^l 2^{m-1-l}}. \tag{4}$$

Solve (4) for $y_k{}^{2^m}$, square both sides, put $y_{m+k}{}^2 = (a_{m+k})/y_k$ as given by the last of equations (3), and we obtain, after a little reduction,

$$y_k{}^{2^{m+1}+(-1)^m} = \prod_{l=0}^{m} a_{k+l}{}^{(-1)^l 2^{m-l}}. \tag{5}$$

We see that (5) is the same as (2) with $m$ replaced by $m+1$. Hence, if (1) holds for $n = m$, it also holds for $n = m+1$. But it has been verified that (1) holds for $n = 1, 2, 3, 4$. Hence it holds for all positive integral values of $n$.

In addition to the two interesting solutions of this problem given above, still another solution is worthy of reproducing, though the author is unknown, having neglected to *attach his name to his work*. If he sent a letter, it became separated from the solution. Will contributors *please follow the printed form* in sending in solutions. Also see full suggestions in October, 1913, issue.

III. Taking the logarithm of each member of each equation, we have the following set of $n$ simultaneous equations in $n$ unknowns.

$$\begin{aligned}
&2\log x_1 + \log x_2 \cdots\cdots\cdots\cdots = \log a_1\\
&\cdots\cdots 2\log x_2 + \log x_3 \cdots\cdots\cdots = \log a_2\\
&\cdots\cdots\cdots 2\log x_3 + \log x_4 \cdots\cdots = \log a_3\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\log x_1 \cdots\cdots\cdots\cdots + 2\log x_n = \log a_n.
\end{aligned}$$

Then, by principles of determinants, we have

$$\log x_1 \begin{vmatrix} 2&1&0&0&\cdots&0\\ 0&2&1&0&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&\cdots&\cdots&2&1\\ 1&0&\cdots&\cdots&0&2 \end{vmatrix} = \begin{vmatrix} \log a_1&1&0&\cdots&0\\ \log a_2&2&1&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ \vdots&&&&\\ \log a_n&0&\cdots&\cdots&2 \end{vmatrix}$$

. . . . . .

. . . . . .

$$\log x_j \begin{vmatrix} 2\ 1\ 0\ \cdots\ 0\ 0 \\ 0\ 2\ 1\ \cdots\ 0\ 0 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ 0\ 0\ 0\ \cdots\ 2\ 1 \\ 1\ 0\ 0\ \cdots\ 0\ 2 \end{vmatrix} = \begin{vmatrix} 2\ 1\ 0\ \cdots\ \log a_1\ 0\ \cdots\ 0 \\ 0\ 2\ 1\ \cdots\ \log a_2\ 0\ \cdots\ 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\ 0\ 0\ \cdots\ \log a_j\ 1\ \cdots\ 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1\ 0\ 0\ \cdots\ \log a_n\ 0\ \cdots\ 2 \end{vmatrix}.$$

Denoting the determinant in the left member of the last equation by $D$ and the one in the right member by $A_j$, we have

$$\log x_j = \frac{A_j}{D},$$

or

$$x_j = e^{\frac{A_j}{D}} \ (j = 1, 2, \cdots, n).$$

Also solved by the PROPOSER, who used logarithms as in the last solution, though he did not use determinants.

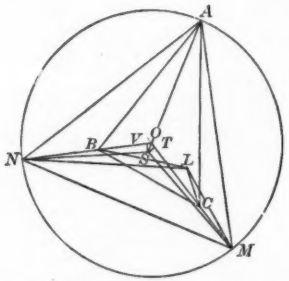
GEOMETRY.

**424. Proposed by H. E. TREFETHEN, Colby College.**

In a given triangle $ABC$, determine by geometric demonstration the point $O$ such that the sum of the distances, $AO + BO + CO$, shall be a minimum.

SOLUTION BY M. E. GRABER, Heidelberg University.

On each of the three sides of the triangle $ABC$, describe a segment of a circle to contain an angle of 120°. The arcs of these circles intersect at a point $O$, about which the angles are each 120°. Then $AO + BO + CO$ is a minimum. Suppose $OA > OB$ or $OC$. Using $OA$ as a radius describe a circumference. Produce $OC$ and $OB$ to meet the circumference at $M$ and $N$. Let $L$ be any other point in $ABC$.




Draw $LC$, $LM$, $LB$, $LN$, $AM$, $MN$, $NA$. Then $AM = MN = NA$ and $LA + LN + LM > OA + ON + OM$. In order to prove this relation, it is evidently sufficient to prove that $OA + ON + OM <$ the sum of the distances from $S$ (the vertex of an isosceles $\Delta$ on $MN$ with the same altitude as $\Delta LMN$)

to $A$, $M$, and $N$. With $MS$ and $NS$ as radii describe arcs cutting $MO$ and $NO$ in $T$ and $V$. Join $T$ and $S$, $V$ and $S$, by straight lines. Then $\angle OVS$ is obtuse and therefore $\angle OSV$ is $< 30°$. When $\angle SOV = 60°$ and $\angle OSV = 30°$, $OV = \frac{1}{2}OS$. When $\angle OSV < 30°$, $OV < \frac{1}{2}OS$ and $OV + OT < OS$. Hence $SM + SN + SA > AO + OM + ON$ and $OM + ON + OA < LM + NL + LA$. Since $LC + CM > LM$ and $LB + BN > LN$, we have $LC + CM + LB + BN + LA > OA + OB + BN + OC + CM$. Hence, $LA + LB + LC > OA + OB + OC$.

Also solved by C. N. SCHMALL, DAVID L. MACKAY, A. M. HARDING and HARVY ROESER.

## CALCULUS.

**336. Proposed by EVA S. MAGLOTT, Ada, Ohio.**

If a right cone stands on an ellipse, prove that its superficial area is $\frac{\pi}{2}(OA + OA')(OA \cdot OA)^{\frac{1}{2}}$ times $\sin\alpha$, where $O$ is the vertex of the cone, $A$ and $A'$ extremities of the major axis of the ellipse, and $\alpha$ is the semi-angle of the cone.

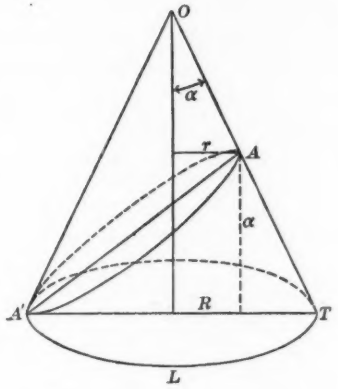
SOLUTION BY ELMER SCHUYLER, Brooklyn, N. Y.

The formula for the surface $AA'LT$ is

$$\frac{\pi}{R - r}\sqrt{a^2 + (R - r)^2}\,\{R^2 - \tfrac{1}{2}(R + r)\sqrt{Rr}\}.$$

(See Finkel's *Mathematical Solution Book*, pp. 318–319).

Here $r = OA \sin\alpha$, $R = OA' \sin\alpha$, and $a = (OA' - OA)\cos\alpha$.




For brevity, put $OA = K$, $OA' = K'$. Then this formula becomes, after slight reduction,

$$\frac{\pi}{\sin\alpha} \cdot \sin^2\alpha\{K'^2 - \tfrac{1}{2}(K + K')\sqrt{KK'}\} = \frac{\pi}{2}\sin\alpha\,\{2K'^2 - (K + K')\sqrt{KK'}\}.$$

The surface $OA'A$ is the total conical surface less the ungula, that is,

$$\frac{\pi}{2}\sin\alpha[2K'^3 - 2K'^3 + (K+K')\sqrt{KK'}] = \frac{\pi}{2}(K+K')(KK')^{\frac{1}{2}}\sin\alpha.$$

The problem rests upon the tedious integration of the form,

$$\int_r^R 2x\cos^{-1}\left[\frac{2Rr-(R+r)x}{(R-r)x}\right]dx.$$

Also solved by A. M. HARDING and P. PENALVER.

A solution of 334 was received from ELMER SCHUYLER, after the the forms were sent to the printer.

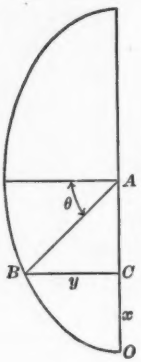
## MECHANICS.

**270. Proposed by W. J. GREENSTREET, Burghfield, England.**

A cycloid has its base vertical. Find the line of quickest descent from the middle point of the base, and its approximate inclination to the horizon.

SOLUTION BY J. SCHEFFER, Hagerstown, Md.

Let $AB$ be any straight line drawn from the middle point $A$ of base of the cycloid, and let $\theta$ be the angle that it makes with the horizon. The time of descent from $A$ to $B$ is $t = \sqrt{\frac{2}{g}\cdot\frac{s}{\sin\theta}}$, $AB$ being $= s$. But $s^2 = (r\pi - x)^2 + y^2$, $x$ and $y$ being subject to the equations of the cycloid, $x = r(\phi - \sin\phi)$, $y = r(1-\cos\phi)$. Substituting, we have $s^2 = r^2[(\pi-\phi)+\sin\phi]^2 + (1-\cos\phi)^2$.




We have to get the minimum of $s/\sin\theta$, which reduces to

$$\frac{(\psi+\sin\psi)^2 + (1+\cos\psi)^2}{\psi+\sin\psi},$$

after putting $\pi - \phi = \psi$ and omitting the constant $r^2$.

Differentiating and setting the differential coefficient equal to zero, we get, after some easy reductions, the transcendental equation $\psi^2 - 2\cos\psi - 2 = 0$. Whence $\psi = \pm 2\cos\psi/2$, and using the positive sign, we find $\psi = 85°$, nearly. Then

$$\tan\theta = \frac{r\pi - x}{y} = \frac{\psi+\sin\psi}{1+\cos\psi}.$$

Hence, finally, $\theta = 53°\ 39'$, nearly.

*Note.* It is assumed in the above solution that the line sought is the *straight line* of quickest descent. Otherwise the problem becomes much more difficult, requiring for its solution the calculus of variations. THE EDITORS.

## NUMBER THEORY.

A solution of 188 was received from ELMER SCHUYLER after the December issue had been made up.

**194. Proposed by L. E. DICKSON, University of Chicago.**

Find two numbers $a$ and $b$ each of two digits such that, when the product $ab$ is found by the usual method, the two partial products to be added are exactly the same as the partial products in getting the product $ba$. Is there a similar pair of numbers of three digits?

SOLUTION BY B. F. YANNEY, University of Wooster.

Let $a = 10x_2 + x_1$, $b = 10y_2 + y_1$.

Then equating corresponding partial products of $ab$ and $ba$, we get, after reducing, $x_1/y_1 = x_2/y_2$ in each case. Hence this is a necessary condition. Obviously, it is also a sufficient condition.

Examples: $a = 21, 32, 43;$
$b = 84, 96, 86.$

In the case of numbers of three digits, let

$$a = 10^2x_3 + 10x_2 + x_1, \quad b = 10^2y_3 + 10y_2 + y_1.$$

Again equating corresponding partial products and reducing, we get

$$10\,x_3y_1 + x_2y_1 = 10\,x_1y_3 + x_1y_2, \tag{1}$$

$$10^2x_3y_2 + x_1y_2 = 10^2x_2y_3 + x_2y_1, \tag{2}$$

$$10\,x_2y_3 + x_1y_3 = 10x_3y_2 + x_3y_1, \tag{3}$$

Since all the digits are less than 10, it is obvious in (2) that $x_3y_2 = x_2y_3$ and $x_1y_2 = x_2y_1$. Whence we get

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3}. \tag{A}$$

Furthermore, $(A)$ substituted in (1) and (3) gives identities as results. Hence $(A)$ is a necessary condition for the numbers $a$ and $b$ to be such as are required. Obviously also $(A)$ is a sufficient condition. Example: $a = 214$, $b = 428$.

In conclusion we state a general theorem, suggested by the above considerations, the proof of which is not difficult.

THEOREM. Two numbers $a$ and $b$ of $n$ digits each are such that, when the products $ab$ and $ba$ are found by the usual method, the corresponding partial products to be added are exactly the same, if and only if the corresponding digits of the two numbers are proportional.

Illustration:

$$\begin{array}{rcr} 1342 & & 2684 \\ 2684 & & 1342 \\ \hline 5368 & = & 5368 \\ 10736 & = & 10736 \\ 8052 & = & 8052 \\ 2684 & = & 2684 \end{array}$$

Solved in similar manner by H. C. FEEMSTER, S. W. REAVES, W. C. EELLS, C. E. GITHENS, A. H. HOLMES, and V. M. SPUNAR.

**195. Proposed by E. B. ESCOTT, University of Michigan.**
Find triangles whose sides are integers and one of whose angles is 60°.

SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

By trigonometry we have for *any* triangle the equation

$$x^2 - 2xy\cos\phi + y^2 = z^2, \tag{1}$$

in which $x$ and $y$ denote the sides including the angle $\phi$, and $z$ the side opposite that angle.

In this equation $\phi$ is supposed to be a given angle and $x, y, z$ wholly unknown and are to be determined in whole numbers.

When $\phi = 60°$, (1) reduces to

$$x^2 - xy + y^2 = z^2. \tag{2}$$

Assuming $z = x - \frac{qy}{p}$ we get from (2), after some reduction,

$$\frac{y}{x} = \frac{p^2 - 2pq}{p^2 - q^2};$$

therefore we may take

$$x = p^2 - q^2 \text{ and } y = p^2 - 2pq.$$

Substituting these values of $x$ and $y$ in the assumed value of $z$ we find

$$z = p^2 - pq + q^2.$$

Solving (2) for $y$ we get

$$y = \tfrac{1}{2}x \pm \sqrt{z^2 - \tfrac{3}{4}x^2}. \tag{3}$$

This result shows that $y$ has two values, and that there are two triangles for each value of $x$ and $z$.

Substutiting the values of $x$ and $y$ found above in (3) we get

$$\begin{aligned} y &= \tfrac{1}{2}(p^2 - q^2) \pm \tfrac{1}{2}(p^2 - 4pq + q^2), \\ &= p^2 - 2pq \text{ or } 2pq - q^2. \end{aligned}$$

It can be shown by a geometrical construction that $y$ has two values.

From what has been done it is obvious that the required triangles for each value of $p$ and $q$ are

$$p^2 - q^2, \quad p^2 - 2pq, \quad p^2 - pq + q^2;$$

and

$$p^2 - q^2, \quad 2pq - q^2, \quad p^2 - pq + q^2;$$

where $p$ can have any value, but $q$ is limited by the following restrictions:

1. $q$ must be less than $p$ and prime to it;

2. $q$ must not be greater than $\frac{1}{2}(p - 1)$ when $p$ is odd, and must be less than $\frac{1}{2}p$ when $p$ is even;

3. $p + q$ must not be divisible by 3, as in that case the sides of the triangle will be divisible by 3.

The following table contains the first seven pairs of such triangles.

| $p$ | $q$ | Sides | | |
|---|---|---|---|---|
| | | $p^2-q^2$ | $p^2-2pq,$ $2pq-q^2$ | $p^2-pq+q^2$ |
| 3 | 1 | 8 | 3 | 7 |
| 3 | 1 | 8 | 5 | 7 |
| 4 | 1 | 15 | 8 | 13 |
| 4 | 1 | 15 | 7 | 13 |
| 5 | 2 | 21 | 5 | 19 |
| 5 | 2 | 21 | 16 | 19 |
| 6 | 1 | 35 | 24 | 31 |
| 6 | 1 | 35 | 11 | 31 |
| 7 | 1 | 48 | 35 | 43 |
| 7 | 1 | 48 | 13 | 43 |
| 7 | 3 | 40 | 7 | 37 |
| 7 | 3 | 40 | 33 | 37 |
| 8 | 3 | 55 | 16 | 49 |
| 8 | 3 | 55 | 39 | 49 |

Also solved by THOMAS E. MASON, A. H. HOLMES, B. F. YANNEY, H. E. TREFETHEN, HORACE OLSON, N. A. DRAXTEN, E. B. ESCOTT, and C. E. GITHENS.

## MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

In the teaching of mathematics there are three fields of experience, now more or less separated, which it is desirable to have joined into one; namely, that of teachers in the better high schools, that of teachers in the smaller colleges and that of teachers in the larger and more powerful institutions. The problems in these fields are not as widely separated or as distinct as some have supposed. We need to find a means of funding the experience from all these three sources so that it may become the common possession of all who are at work in these fields.

It is our desire that this department of the MONTHLY shall become an effective means to this end. For this we need the coöperation of a large number of persons engaged in the teaching of mathematics. In this connection there are at least three ways in which our readers may render valuable service to the cause of mathematical education:

(*a*) Give us suggestions as to the plans along which this department should be developed.

(*b*) Propose specific questions on which it is desirable to know what is the general opinion or what is the experience of individual teachers.

(*c*) Respond promptly to the questions printed with a statement of your opinion or an account of your experience or both.

Already we have had a gratifying expression of interest in these matters. We have in hand now an excellent answer to question 5 on culture mathematics and an excellent answer to question 6 on vocational mathematics. Let us have further replies to these and to other questions; let us also have other useful questions proposed. If the interest invoked and the amount of good matter received requires it, we can enlarge this department in later issues so as to meet the demand made upon it.

QUESTIONS.

**7.** What place should be given to the history of mathematics in courses for prospective high school teachers, and why?

**8.** One of our correspondents would like to know the experiences of other teachers in giving practice teaching in mathematics during a college course. May we have here the experiences of several teachers?

**9.** What is the present state of experience with coördinated courses in high school mathematics? What contribution does this promise to the development of mathematics teaching in high schools? What about the corresponding matters in college mathematics? (Note.—An individual correspondent need not answer all the questions in number 9; it is sufficient if he answers only one.)

REPLIES.

Instead of here giving replies to questions on which no communication has yet been printed, we return this time to question 2, giving our space to the following interesting note which was called forth in connection with that question. In this connection it will be of interest to many of our readers to know that the text-book, "The Principles of Projective Geometry," by J. L. S. Hatton, to which reference was made by Mr. Stromquist in the January issue, will be reviewed by h m in the MONTHLY in the near future.

## A NOTE ON SYNTHETIC PROJECTIVE GEOMETRY.

By LAO G. SIMONS, Normal College of the City of New York.

The writer of this note is in hearty sympathy with the article by Professor Bussey in the AMERICAN MATHEMATICAL MONTHLY for November, 1913, on "Synthetic Projective Geometry as an Undergraduate Study," and ventures the hope that the experience of a college teacher who has for five years conducted courses simultaneously in synthetic projective geometry and methods of teaching secondary mathematics may be of interest to other teachers of mathematics.

The course in projective geometry in the Normal College is a three-hour course for one semester and follows a set of notes prepared by Dr. George H. Ling, formerly of Columbia University and now at the University of Saskatchewan, based on Doehlemann's Projektive Geometrie in the *Sammlung Göschen*. It includes the following topics in the order named: Part I. Central projection and projection from an axis with the elements at infinity, geometric prime forms, perspective prime forms; principle of duality; the anharmonic ratio; harmonic forms defined metrically and descriptively; complete four-point and four-side with examples of such interest as bisecting a line or an angle by means of a straight edge alone; figures in plane homology; projective relation between prime forms with the geometric construction of the fourth element of an anharmonic ratio, similar and congruent ranges, with such examples as determining a line through the inaccessible intersection of two lines; superposed projective forms with the important theorems on the number of double elements possible and Steiner's geometric determination of these, involution, Desargues's theorem for a complete quadrangle, brought up again later for the circle and conic. Part II. Applications. Geometrical figures (in the plane) generated by projective prime forms,

loci generated by intersections of projective flat pencils, with the problem to find the intersections (if any) of a line with the curve of the second order determined by five points, and the correlative, envelope of a line which moves so as always to join corresponding points of projective ranges, and the problem to find the tangents (if any) through any given point to any curve of the second class which is determined by any five given lines (In passing—Are not this locus and this envelope good subjects for a moving picture?), connection between curves of the second order and curves of the second class; poles and polars, their properties, the self-polar triangle and certain theorems leading to the establishment of the validity of the principle of duality for figures in one plane (concerning which there is some discussion), poles and polars of the elements at infinity. The notes include a chapter on geometrical figures in the sheaf generated by projective prime forms, but the time has been too limited to touch upon this part of the subject. The text includes many exercises, among them theorems necessary to the sequence of the work, an excellent plan for the development of the minds of prospective teachers.

Our pupils are not allowed to take the projective geometry before the end of the second year in college and most of them elect it even later. They are taking as a parallel course the second semester of analytic geometry or they have finished this work, which includes the solid analytic geometry. Only pupils who are really interested in mathematics are advised to take this course. Indeed, a poor student is so strongly "advised" against it that she decides for herself to elect something else. The work is conducted with almost no so-called class recitations. It is assumed that a girl can keep up to the mark so that the possible answering of a few questions at the beginning of the hour enables the class to go right ahead with new work.

And the girls enjoy the work. Term after term, the experience is the same. From the first lesson right through to the last, there is no abatement of interest. The students uniformly do a high order of work in the course, and the rating is the result of four tests during the semester taken in connection with the judgment of the instructor on the individual girl. The opinions of the students expressed to me are that the course has been an inspiration and repeatedly there has been the wish that it might have extended through the year. They feel that the course has given to them the great enjoyment that a mathematics student experiences when he has arrived at a definite conclusion through steps of pure reasoning, when he has proved a fact true that was not at first apparent. It is encouraging that the purely abstract reasoning with no attempt at possible uses for the subject matter gives such pleasure to the college student. The naïve statement of one student was: "The study of projective geometry came nearer to my idea of a recreational course in mathematics than any of the other courses." Personally, I find no more intense interest in any work that I present and no work brings me such real joy and satisfaction.

My judgment of the course is that it broadens, as few college subjects do, the minds of prospective high school teachers. The method of approach is so

different from that of the analytic geometry that the pupils gain at a bound a sense of the variety of mathematical treatment. After having defined a conic as the locus of a point that moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line, after having translated this property into algebraic language, and after having studied the properties of the conic from this equation, it is an illuminating step to find that a conic is the locus of the intersections of corresponding rays of projective flat pencils, that three sets of lines, properly related to each other, may be used to construct these curves that play so important a part in the physical world, and that these curves may be studied from this point of view.

Just one illustration will serve to show what points of contact there are with elementary geometry. In triangles in homology, when the center of homology is at an infinite distance in the direction perpendicular to the axis of homology, and again when the axis of homology is at an infinite distance, the constant in both cases being $-1$, the two figures are congruent, in reverse order in the first case, and in the same order in the second. These very elements at infinity are a part of the broadening influence of this subject. They may and should be introduced in the analytic geometry but they *must* be employed in the projective geometry; and thus the satisfaction comes in finding theorems that are true under all conditions.

It may be well in closing to recall the report of the Committee of Ten on secondary school studies. "A place should also be found either in the school or college course for at least the elements of the modern synthetic or projective geometry. It is astonishing that this subject should be so generally ignored, for mathematics offers nothing more attractive. It possesses the concreteness of the ancient geometry, without the tedious particularity, and the power of analytical geometry without the reckoning, and by the beauty of its ideas and methods illustrates the esthetic quality which is the charm of the higher mathematics, but which the elementary mathematics in general lacks."

## NOTES AND NEWS.

UNDER THE DIRECTION OF FLORIAN CAJORI.

Mr. W. C. EELLS, head of the department of mathematics in Whitworth College, Washington, has been elected instructor in the United States Naval Academy at Annapolis.

The January number of the *Hibbert Journal* contains a short article by Professor C. W. COBB, of Amherst College, on "Certainty in Mathematics and in Theology."

Mr. HYLAND CLAIR KIRK has an article on "The fourth dimension" in the December number of the *Open Court*. It is a humorous satire.

BENJAMIN O. PEIRCE, Hollis professor of mathematics and natural philosophy in Harvard college, died at Cambridge, Mass., January 15, 1914. He was a

member of learned societies and the author of several books on mathematics and physics.

Mr. A. L. MILLER, A.B. 1911, A.M. 1913, of Harvard, for two years instructor in mathematics at Harvard, is now instructor in mathematics at the University of Michigan.

The mathematical club at the University of Minnesota is taking up the study of the fundamental existence theorems given by Professor G. A. BLISS in the Princeton Colloquium Lectures recently published by the American Mathematical Society.

The January number of *School Science and Mathematics* contains two articles that are of interest to readers of the MONTHLY. They are "Suggestions for the prospective mathematician," by G. A. MILLER, and "Needed,—a funeral of algebraic phraseology," by EFFIE GRAHAM, of the high school at Topeka, Kansas.

An answer-book to the DAVIS-BRENKE Calculus has just been issued by the Macmillan Company. As such an answer-book saves much of the instructor's time, it adds to his efficiency as a teacher.

A review of Paolo Ruffini's researches, carried on during the years 1799–1813, on the impossibility of the algebraic solution of the quintic and on groups of operations, is given by E. BORTOLOTTI in the *Memorie della R. Accad. di Scienze etc. in Modena*, S. 3, Vol. 12 (Sezione scienze). He aims to give some of the results reached by Ruffini in order to remove some misconceptions that are prevalent relating to Ruffini's work and in order to encourage students to read Ruffini's original publications.

*Nature* (Dec. 18, 1913) contains a review of A. N. WHITEHEAD and B. RUSSEL'S *Principia Mathematica*, Vol. III, 1913. This volume treats of the theory of series and the theory of measurement. As in the preceding volumes, so here the authors use a new symbolism and deal with the logical deduction of the propositions from merely logical foundations.

The engineering departments of Harvard University and Massachusetts Institute of Technology are to be combined, according to a press dispatch, the departments to be conducted in the buildings of the Institute now being erected in Cambridge. The president of the Institute will be the executive head of the work, and the faculty will consist of the faculty of the Institute, enlarged by the addition of Harvard professors in the departments involved. The two institutions are to remain unchanged in name, organization, and title to property, the Institute of Technology to furnish the buildings, laboratories, and the two institutions to furnish jointly equipment and contributions to the necessary funds.

At the November meeting of the Association of Teachers of Mathematics in the Middle States and Maryland the following papers were read: MAURICE J. BABB and CHARLES F. WHEELOCK, "Are particular abilities necessary for the pupil to gain an understanding of the elementary and secondary mathematics as usually given at the present time"; JAMES C. BROWN, "A comparison at equal

school ages of the attainments in mathematics of the European and American schoolboy, with a consideration of causes and remedies"; ALBERT D. YOCUM, "Mathematics as a means to culture and discipline"; ROMIETT STEVENS, "The use of the question in the classroom."

During the recent meeting of the American Association for the Advancement of Science, held at Atlanta, Georgia, Professor H. S. WHITE of Vassar College was elected chairman of Section A. The retiring chairman of this Section, Professor E. B. VAN VLECK of Wisconsin University, delivered an address entitled "The influence of Fourier's series upon the development of mathematics." The next meeting of this Association will be held in Philadelphia, during the convocation week, 1914–15.

According to the *Bulletin des Sciences Mathématiques*, December, 1913, page 355, the Minister of Public Instruction of France began to study the question of requiring "candidats à l'agrégation des sciences mathématiques" to pass an examination in the elements of the theory of groups and their applications in the theory of equations.

During the public meeting of the Paris Academy of Sciences, held December 15, 1913, it was announced that the "Grand Prix" (3,000 francs) for the year 1916 would be given for a work applying the methods of POINCARÉ to the integration of some linear algebraic differential equations. The Academy reserves the right to examine published papers as well as manuscripts in connection with this prize.

At the same meeting of the Paris Academy of Sciences especial attention was called to the important work by M. SUNDMANN, a young astronomer of Finland, towards the solution of the famous problem of three bodies. The Academy awarded him the Pontécoulant prize, the amount of the award being doubled (1,400 francs). The recommendation was made by a commission of which EMILE PICARD was chairman. The statement of the problem of three bodies is well-known: To find the paths described, and the velocities at each point, of three bodies which attract one another according to Newton's law, it being supposed that the bodies are reduced to material points devoid of extension. Sundmann's research is along the path first blazed by Painlevé, continued by Levi-Civita and others. It is interesting to recall that in 1772 this Academy awarded a prize for an essay on the problem of three bodies to the illustrious LAGRANGE. An illuminating historical review of the problem of several bodies was given some years ago by President Edgar O. Lovett of the Rice Institute in Houston, Texas. It is published in *Science*, N. S., Vol. 29, 1909, pp. 81–91.

Copies of the January, 1913, issue of the MONTHLY are *very much needed*. Any one having an extra copy of this number will confer a great favor by sending it to the MANAGING EDITOR at once.

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# A NOTE ON PLANE KINEMATICS.

By ALEXANDER ZIWET AND PETER FIELD, University of Michigan.

## INTRODUCTION.

It appears from the nature of the subject that the use of vector methods must be of advantage in studying the motion of a rigid body. A. FÖPPL'S *Technische Mechanik*, F. CASTELLANO'S *Meccanica razionale* (1894), and especially R. MARCOLONGO'S *Meccanica razionale* (Milano, Hoepli, 1905, 2 vols., German edition by TIMERDING, Teubner, 1911–12) bear evidence to this effect.

The present paper is confined to *plane* motion where the number of new symbols is very small. The notation used is that of C. BURALI-FORTI et R. MARCOLONGO, *Eléments de calcul vectoriel*, Paris, Hermann, 1910.

The vector from the point $O$ to the point $P$ is denoted by $P - O$, or by a single bold-faced letter such as $\boldsymbol{p}$. The time-derivative of a moving point $P$ is denoted by a dot; thus, if $P$, $P'$ are the positions of the point at the times $t$, $t + \Delta t$, we have

$$\dot{P} = \lim_{\Delta t = 0} \frac{P' - P}{\Delta t}$$

for the *velocity* of $P$. The *acceleration* of $P$ is the second time-derivative $\ddot{P}$. The time-derivative of the vector $\boldsymbol{p} = P - O$ is $\dot{\boldsymbol{p}} = \dot{P} - \dot{O}$; this is the velocity of $P$ *relative* to $O$.

The absolute value, or length, of a vector $\boldsymbol{p} = P - O$ is indicated by mod $\boldsymbol{p}$, or mod $(P - O)$. The scalar product of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, i. e., the product of their lengths into the cosine of the angle of their directions, is written $\boldsymbol{a} \times \boldsymbol{b}$; for $\boldsymbol{a} \times \boldsymbol{a}$ we write $\boldsymbol{a}^2$. The vector product $\boldsymbol{a} \wedge \boldsymbol{b}$ (read $\boldsymbol{a}$ vec $\boldsymbol{b}$), which is the vector at right angles to both $\boldsymbol{a}$ and $\boldsymbol{b}$, representing by its length the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$, is not required in the plane.

Rotation through a right angle in the positive (say, counter-clockwise) sense is indicated by $i$; thus $i\boldsymbol{a}$, $i\dot{P}$ are the vectors obtained by turning $\boldsymbol{a}$, $\dot{P}$ through